AMALGAMATED ALGEBRA EXTENSIONS DEFINED BY VON NEUMANN REGULAR AND SFT CONDITIONS

KHALID LOUARTITI AND NAJIB MAHDOU

Abstract. Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. In this paper, we characterize $R \bowtie^f J$ to be Von Neumann regular ring and SFT ring, respectively.

1. Introduction

Throughout this paper all rings are assumed to be commutative with identity element and the dimension of a ring means its Krull dimension.

Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A with B along J with respect to f* (introduced and studied by D'Anna, Finacchiaro, and Fontana in [9, 10]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [11, 12, 13]) and denoted by $A \bowtie I$. Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([9, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealization (cf. [16, page 2]), and the CPI extensions (in the sense of Boisen and Sheldon [5]) are strictly related to it ([9, Example 2.7 and Remark 2.8]).

On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by Anderson in [1] and motivated by a classical construction due to Dorroh [8], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [9, Section 2]. Also, the authors consider the iteration of the amalgamation process, giving some geometrical applications of it.

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [9, Section 4]. This point of view allows the authors in [9, 10] to provide an ample

2000 *Mathematics Subject Classification.* 13D05, 13D02.

Key words and phrases. Von Neumann regular ring, SFT ring, amalgamated algebras along an ideal.

description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [9], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [10], they pursue the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

Recall that a ring R is called *Von Neumann regular* if for each $a \in R$, there exists $x \in R$ such that $axa = a$. It is proved in [6, Theorem 2.1] that, for an ideal I of R, $R \bowtie I$ is Von Neumann regular if and only if R is Von Neumann regular. In section 2, we extend this result to amalgamated algebra along an ideal.

An ideal I is called an SFT-ideal if there exists a naturel number k and a finitely generated ideal $J \subseteq I$ such that $a^k \in J$ for each $a \in I$. An SFT ring is a ring in wich every ideal is an SFT-ideal.
In [2], Arnold studies the Krull dimension of a power series ring $R[[x]]$ over a ring R and showed that the dimension is infinite unless R is an SFT ring, which forces us to consider only SFT rings when we study finite-dimensional power series extensions.

For any ring A with finite Krull dimension, we have:
A Noetherian $\Longrightarrow dimA[[X]] < \infty \Longrightarrow A$ SFT ring.

One important family of SFT rings is that of SFT Prüfer domains, which are also called generalized Dedekind domains. The beautiful discovery of Arnold is that, for D a finite-dimensional SFT-Prüfer domain, $dimD[[x_1, ..., x_n]] = n(dimD) + 1$, and so $D[[x_1, ..., x_n]]$ is an SFT ring [3]. In 2007, Kang and Park [15, Theorem 10] extend Arnold's result to the infinite-dimensional case, thus proving that over an infinite-dimensional SFT Prüfer domain D, the power series ring $D[[x_1, ..., x_n]]$ is an SFT ring. In 2010, Park [17, Theorem 2.4] shows that, if R is an m-dimensional SFT globalized pseudo-valuation domain, then $dimR[[x_1, ..., x_n]] = mn + 1$ or $mn + n$.

SFT rings are similar to Noetherian rings and they have many nice properties. It had been a long-standing open question if the power series extension of an SFT ring is also an SFT ring. Coykendall's counterexample to this appears in [7]. Remark that these rings are coherent. Coykendall prove also that a ring R is SFT if and only if each prime ideal is SFT ([7]).

In this work, we characterize $R \bowtie^f J$ to be a Von Neumann regular ring and SFT ring, respectively. Our results generate new and original examples which enrichy the current literature with new families of Von Neumann regular rings and SFT rings.

2. Von Neuman regular amalgamated algebra along an ideal

This section characterize the amalgamated algebra along an ideal $R \bowtie^f J$ to be a Von Neumann regular ring. The main result (Theorem 2.1) enriches the literature with original examples of Von Neumann regular rings.

Theorem 2.1. *Let A and B be two rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. Then, $A \bowtie^f J$ is a Von Neumann regular ring if and only if the following statements holds:*

(1) *A is a Von Neumann regular ring.*
(2) $\mathrm{Nilp}\,(B) \cap J = \{0\}$.
(3) *Every prime ideal of B which don't contains J is maximal.*

Proof. For each ideals P and Q of A and B respectively, set $P'^f := P \bowtie^f J := \{(p, f(p) + j) \mid p \in P, j \in J\}$ and $\overline{Q}^f := \{(a, f(a) + j) \mid a \in A, j \in J, f(a) + j \in Q\}$.

Assume that $R \bowtie^f J$ is a Von Neumann regular ring. Then, it is reduced. Hence, by [9, Proposition 5.4], A is reduced and $\mathrm{Nilp}\,(B) \cap J = \{0\}$. Let P be a prime ideal of A. Then, by [10, Proposition 2.6], P'^f is a prime ideal of $R \bowtie^f J$. Hence, it is maximal since $A \bowtie^f J$ is Von Neumann regular. Consequently, by [10, Proposition 2.6], P is a maximal ideal of A. Hence, A is a Von Neumann regular ring. Thus, (1) and (2) hold. Let Q be a prime ideal of B not containing J. By [10, Proposition 2.6], $\overline{Q}^f$ is a prime ideal of $A \bowtie^f J$, and so maximal. Then, also by [10, Proposition 2.6], Q is a maximal ideal of B. Hence, (3) holds.

Conversely, suppose that (1), (2) and (3) hold. By [9, Proposition 5.4], the statements (1) and (2) imply that $A \bowtie^f J$ is reduced. Moreover, from [10, Proposition 2.6 (3)], $\mathrm{Spec}(A \bowtie^f J) = \{P'^f \mid P \in \mathrm{Spec}(A)\} \cup \{\overline{Q}^f \mid Q \in \mathrm{Spec}(B), I \not\subset J\}$ and $\mathrm{Max}(A \bowtie^f J) = \{P'^f \mid P \in \mathrm{Max}(A)\} \cup \{\overline{Q}^f \mid Q \in \mathrm{Max}(B), I \not\subset J\}$. Since A is Von Neumann regular, then $\mathrm{Spec}(A) = \mathrm{Max}(A)$. On the other hand, (3) means that $\{Q \in \mathrm{Spec}(B), I \not\subset J\} = \{Q \in \mathrm{Max}(B), I \not\subset J\}$. Hence, $\mathrm{Spec}(A \bowtie^f J) = \mathrm{Max}(A \bowtie^f J)$. Consequently, $A \bowtie^f J$ is Von Neumann regular, as desired. □

Remark 2.2. If A is Von Neumann regular ring and I is an ideal of A then $\mathrm{Nilp}\,(A) \cap I = \{0\} \cap I = \{0\}$ and every prime ideal (in particular these which doesn't contains I) is maximal. Hence, Theorem 2.1 is clearly a generalization of [6, Theorem 2.1.].

Corollary 2.3. *Let A and B be two rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. If A and B are both Von Neumann regular rings then so is $A \bowtie^f J$.*

Proof. Follows immediately from Theorem 2.1 □

Recall that a ring R is called Boolean ring if $x^2 = x$ for each $x \in R$. Boolean rings are Von Neumann regular.

Example 2.4. Consider the ring $B = \prod_{i=1}^{n} K_i$ with $K_i = \{0; 1\}$ and A the subring of stationary sequences of B. Set $J = \bigoplus_{i=1}^{n} K_i$ which is an ideal of B, and let $\iota : A \longrightarrow B$ be the canonical embedding of A into B. Then $A \bowtie^{\iota} J$ is a Von Neumann regular ring.

Proof. Follows from Corollary 2.3 since B and A are both Boolean rings, and then Von Neumann regular rings. $\square$

It is well known that semisimple rings coincide with Noetherian Von Neumann rings. Hence, we have the following corollary.

Corollary 2.5. *Let A and B be two rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. Then, $A \bowtie^{f} J$ is a semisimple ring if and only if the following statements hold:*

(1) *A is a semisimple ring.*
(2) $\mathrm{Nilp}\,(B) \cap J = \{0\}$.
(3) *Every prime ideal of B which doesn't contains J is maximal.*
(4) *$f(A) + J$ is a Noetherian ring.*

In particular, if A and B are both semisimple and the ring homomorphism $\overline{f} : A \to B/J$ is finite, then $A \bowtie^{f} J$ is semisimple.

Proof. By [9, Proposition 5.6], $A \bowtie^{f} J$ is Noetherian if and only if A and $f(A)+J$ are Noetherian. Then, the desired equivalence follows directly from Theorem 2.1.

The last particular statement follows from [9, Proposition 5.8] and Corollary 2.3. $\square$

3. SFT amalgamated algebra along an ideal

The main result of this section characterize the amalgamated algebra along an ideal $R \bowtie^{f} J$ to be an SFT ring. This result (Theorem 3.1) enriches the literature with original examples of SFT rings.

Theorem 3.1. *Let A and B be two rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. Then, $A \bowtie^{f} J$ is an SFT ring if and only if A and $f(A) + J$ are both SFT rings.*

The proof of the theorem involves the following lemmas of independent interest.

Lemma 3.2. *Let R be a ring and K be a proper ideal of R. If R is an SFT ring then so is R/K.*

Proof. Let $\mathcal{J}$ be an ideal of R/K. There exists an ideal J of R such that $\mathcal{J} = \overline{J}$. Since R is an SFT ring there exists a finitely generated ideal I of R

and a positive integer k such that $I \subset J$ and $x^k \in I$ for each $x \in J$. Thus, $\overline{I}$ is a finitely generated ideal of R/K, $\overline{I} \subset \mathcal{J}$ and $\overline{x}^k \in \overline{I}$ for each $\overline{x} \in \mathcal{J}$. Hence, R/K is an SFT ring, as desired. □

Lemma 3.3. *Let R be a ring. If I and J are two SFT ideals of R then so is $I + J$.*

Proof. Assume that I and J are SFT ideals of R. Then, there exists finitely generated ideals I' and J' and two positive integers k and k' such that $I' \subset I$, $J' \subset J$, $x^k \in I'$ for each $x \in I$ and $y^{k'} \in J'$ for each $y \in J$. Clearly, $I' + J'$ is a finitely generated subideal of $I + J$. Moreover, for each $x \in I$ and $y \in J$, we have

$$\begin{aligned}(a+b)^{k+k'} &= \sum_{i=0}^{i=k+k'} C^i_{k+k'} a^i b^{k+k'-i} \\ &= [\sum_{i=0}^{i=k} C^i_{k+k'} a^i b^{k-i}] b^{k'} + [\sum_{i=k+1}^{i=k+k'} C^i_{k+k'} a^{i-k} b^{k+k'-i}] a^k\end{aligned}$$

with $C^i_{k+k'} = \dfrac{(k+k')!}{i!(k+k'-i)!}$. Hence, $(a+b)^{k+k'} \in I' + J'$. Consequently, $I + J$ is an SFT ideal of R. □

Lemma 3.4. *Let A and B be two rings, J an ideal of B, $f : A \to B$ be a ring homomorphism and let I be an ideal of A. If $I \bowtie^f J$ is an SFT ideal of $A \bowtie^f J$ then I is an SFT ideal of A with equivalence if J is an SFT ideal of $f(A) + J$.*

Proof. For a ring R, we denote by $L := < a_i \mid i = 1, .., n >_R$ the finitely generated ideal of R generated by $a_1, a_2, .., a_n$.

Assume that $I \bowtie^f J$ is an SFT ideal of $A \bowtie^f J$. Then, there exists finitely generated ideal $K := < (i_l, f(i_l) + j_l) \mid l = 1, .., n >_{A\bowtie^f J}$ of $A \bowtie^f J$ and a positive integer k such that $K \subset I \bowtie^f J$ and $x^k \in K$ for each $x \in I \bowtie^f J$. Set $I' = < i_l \mid l = 1, .., n >_A$. It is clear that $I' \subset I$ and let $i \in I$. Since $(i, f(i)) \in I \bowtie^f J$, we get $(i^k, f(i^k)) = (i, f(i))^k \in K$. Thus, $i^k \in I'$. Hence, I is an SFT ideal of A.

Assume that J is an SFT ideal of $f(A) + J$. Then there exists a finitely generated ideal $J' = < j_l \mid k = 1, .., m >_{f(A)+J}$ of $f(A) + J$ and a positive integer k such that $j^k \in J'$ for each $j \in J$. Set $\overline{J'} := < (0, j_l) \mid i = 1, .., m >_{A\bowtie^f J}$. On the other hand, I is an SFT ideal of A. Then, there exists a finitely generated ideal $I' = < i_l \mid l = 1, .., n >_A$ of A and a positive integer k' such that $I' \subset I$ and $i^{k'} \in I'$ for each $i \in I$. Set $\overline{I'}^f := < (i_l, f(i_l)) \mid l = 1, .., n >_{A\bowtie^f J}$. Clearly, $K := \overline{I'}^f + \overline{J'}$ is a finitely generated ideal of $A \bowtie^f J$ and $K \subset I \bowtie^f J$. Moreover, for each $(i, f(i) + j) \in A \bowtie^f J$, $(i, f(i) + j) = (i, f(i)) + (0, j)$ and $(i, f(i))^k \in \overline{I'}^f$ since $i^k \in I'$ and $(0, j)^{k'} \in$

$\overline{J'}$ since $j^{k'} \in J'$. Hence, as in the proof of Lemma 3.3, we can prove that $(a, f(a)+j)^{k+k'} \in K$. Consequently, $I \bowtie^f J$ is an SFT ideal of $A \bowtie^f J$. □

Proof of Theorem 3.1. Assume that $A \bowtie^f J$ is an SFT ring. By [9, Proposition 5.1 (3)], the rings A and $f(A)+J$ are homomorphic images of $A \bowtie^f J$. Then, using Lemma 3.2, they are SFT rings.

Conversely, for each prime ideals P and Q of A and B respectively, set $P'^f := P \bowtie^f J := \{(p, f(p)+j) \mid p \in P, j \in J\}$ and $\overline{Q}^f := \{(a, f(a)+j) \mid a \in A, j \in J, f(a)+j \in Q\}$. Let P be a prime ideal of A. Then, by [10, Proposition 2.6], P'^f is a prime ideal of $R \bowtie^f J$. Hence, by Lemma 3.4, it is an SFT ideal of $A \bowtie^f J$. Let $\overline{Q}^f$ be a prime ideal of $A \bowtie^f J$, then $Q_0 = \overline{Q \bigcap (f(A)+J)}$ is an ideal of $(f(A)+J)/J$. Hence, there exists a finitely generated ideal $Q'_0 =< \overline{(a_i, f(a_i)+j_i)} \mid i = 1,..,n >_{(f(A)+J)/J}$ of $(f(A)+J)/J$ and a positive integer k_0 such that $Q'_0 \subset Q_0$ and $x^{k_0} \in Q'_0$ for each $x \in Q_0$. Set $L_0 =< (a_i, f(a_i)+j_i) \mid i = 1,..,n >_{A \bowtie^f J}$. Then $I = f^{-1}(J) \bigcap P_A(\overline{Q}^f) := \{a \in A \mid f(a) \in J \ ; \ \exists j \in J \mid f(a)+j \in Q\ \}$ is an ideal of A, and so there exists a finitely generated ideal $I' =< a_i \mid i = n+1,..,m >$ of A, and a positive integer k_1 such that $I' \subset I$ and $x^{k_1} \in I'$ for each $x \in I$. Set $L_1 =< (a_i, f(a_i)+j_i) \mid i = n+1,..,m >_{A \bowtie^f J}$.
Or $Q_1 = Q \bigcap J$ is an ideal of $f(A)+J$. Since $f(A)+J$ is an SFT ring, then there exists a finitely generated ideal $Q'_1 =< j_i \mid i = m+1,..,l >_{f(A)+J}$ of $f(A)+J$ and a positive integer k_2 such that $Q'_1 \subset Q_1$ and $x^{k_2} \in Q'_1$ for each $x \in Q_1$. Set $L_2 =< (0, j_i) \mid i = m+1,..,l >_{A \bowtie^f J}$ and $L = L_0 + L_1 + L_2$. Let $(a, f(a)+j) \in \overline{Q}^f$, then $\overline{(f(a)+j)}^{k_0} = \sum_{i=1}^{m} \overline{(f(a_i)+j_i)(f(b_i)+j'_i)}$.

Set $\beta = (f(a)+j)^{k_0} - \sum_{i=1}^{m} (f(a_i)+j_i)(f(b_i)+j\ '_i) \in J$. Then $f(a^{k_0} - \sum_{i=1}^{m} a_i b_i) \in J$. Hence $\alpha = a^{k_0} - \sum_{i=1}^{m} a_i b_i \in f^{-1}(J)$. Therefore,

$$
\begin{aligned}
(a, f(a)+j)^{k_0} &= (\alpha + \sum_{i=1}^{m} a_i b_i, \beta + \sum_{i=1}^{m} (f(a_i)+j_i)(f(b_i)+j'_i)) \\
&= \sum_{i=1}^{m} (a_i, f(a_i)+j_i)(b_i, f(b_i)+j\ '_i) + (\alpha, \beta).
\end{aligned}
$$

Since $(a, f(a)+j)^{k_0} \in \overline{Q}^f$, then $C_1 = \sum_{i=1}^{m} (a_i, f(a_i)+j_i)(b_i, f(b_i)+j\ '_i) \in \overline{Q}^f$. Consequently, $(\alpha, \beta) \in \overline{Q}^f$. Therefore, $(\alpha, \beta) = (\alpha, f(\alpha)+e)$ such that $e \in J$ and $f(\alpha)+e \in Q$.

Then $\alpha \in I$ and $\alpha^{k_1} = \sum_{i=n+1}^{m} a_i a_i\,'$. Thus,

$$\begin{aligned}
(\alpha,\beta)^{k_1} &= (\alpha, f(\alpha)+e)^{k_1} = (\alpha^{k_1}, (f(\alpha)+e)^{k_1}) = (\alpha^{k_1}, f(\alpha)^{k_1} + e\,'')\\
&= (\sum_{i=n+1}^{m} a_i a_i\,', \sum_{i=n+1}^{m} f(a_i)f(a_i\,') + e\,'')\\
&= (\sum_{i=n+1}^{m} a_i a_i\,', \sum_{i=n+1}^{m} (f(a_i)+j_i)f(a_i\,') + e\,')\\
&= \left[\sum_{i=n+1}^{m} (a_i, f(a_i)+j_i)(a_i\,', f(a_i\,')\right] + (0, e\,').
\end{aligned}$$

Since $(\alpha,\beta) \in \overline{Q}^f$; $C_2 = \sum_{i=n+1}^{m} (a_i, f(a_i)+j_i)(a_i\,', f(a_i\,')\} \in \overline{Q}^f$. Then, $(0, e\,') \in \overline{Q}^f$ and $e\,' \in Q_1$. Therefore, $e'\,^{k_2} = \sum_{i=m+1}^{f} (f(b_i)+e_i)j_i$. Hence, $(0, e\,')^{k_2} = \sum_{i=m+1}^{f} (b_i, f(b_i)+e_i)(0, j_i) \in L_2$.
Consequently,

$$\begin{aligned}
(a, f(a)+j)^{k_0+k_1+k_2} &= \Big[(a, f(a)+j)^{k_0}\Big]^{k_1+k_2}\\
&= \left[\sum_{i=1}^{m}(f(a_i)+j_i)(f(b_i)+j\,'_i) + (\alpha,\beta)\right]^{k_1+k_2}\\
&= \Big[[C_1 + (\alpha,\beta)]^{k_1}\Big]^{k_2}\\
&= \left[\sum_{t=0}^{k_1} \begin{pmatrix} t \\ k_1 \end{pmatrix} (C_1)^t (\alpha,\beta)^{k_1-t}\right]^{k_2}\\
&= \left[\sum_{t=0}^{k_1-1} \begin{pmatrix} t \\ k_1 \end{pmatrix} (C_1)^t (\alpha,\beta)^{k_1-t} + (\alpha,\beta)^{k_1}\right]^{k_2}\\
&= \left[\sum_{t=0}^{k_1-1} \begin{pmatrix} t \\ k_1 \end{pmatrix} (C_1)^t (\alpha,\beta)^{k_1-t} + C_2 + (0, e\,')\right]^{k_2}\\
&= \left[\sum_{v=0}^{k_2-1} \begin{pmatrix} v \\ k_2 \end{pmatrix} \left[\sum_{t=0}^{k_1-1} \begin{pmatrix} t \\ k_1 \end{pmatrix} (C_1)^t (\alpha,\beta)^{k_1-t} + C_2\right]^v (0, e\,')^{k_2-v}\right]\\
&\quad + (0, e\,')^{k_2}.
\end{aligned}$$

But $\left[\sum_{t=0}^{k_1-1}\begin{pmatrix} t \\ k_1 \end{pmatrix}(C_1)^t(\alpha,\beta)^{k_1-t}+C_2\right]^v(0,e\,')^{k_2-v}\in L_0+L_1$, and $(0,e\,')^{k_2}\in L_2$. Hence $(a,f(a)+j)^{k_0+k_1+k2}\in L_0+L_1+L_2$ and so $\overline{Q}^f$ is an SFT ideal of $A\bowtie^f J$. Consequently, $A\bowtie^f J$ is an SFT ring. □

The following two Corollaries is an immediate consequence of Theorem 3.1 and Lemma 3.2.

Corollary 3.5. *Let A be a ring, I be an ideal of A, J be an ideal of $B := A/I$ and let $f : A \to B(= A/I)$ be the canonical homomorphism. Then, $A\bowtie^f J$ is an SFT ring if and only if so is A.*

Corollary 3.6. *Let A be a ring and I be an ideal of A. Then, $A\bowtie I$ is an SFT ring if and only if so is A.*

Noetherian rings are both SFT and coherent rings. In [4, Page 344], Bakkari gives examples of non-coherent SFT-rings. Now, we are able to give new examples of non-coherent SFT rings.

Example 3.7. Let A be a non-coherent SFT ring, I be an ideal of A, J be an ideal of $B := A/I$ and let $f : A \to B(= A/I)$ be the canonical homomorphism. Then:

(1) $A\bowtie^f J$ is an SFT ring.
(2) $A\bowtie^f J$ is not coherent.

Proof. 1) By Corollary 3.5 since A is an SFT ring.
2) $A\bowtie^f J$ is not coherent by [14, Theorem 4.1.5] since A is a module retract of $A\bowtie^f J$ and A is not coherent. □

Example 3.8. Let A be a non-coherent SFT ring and I be an ideal of A. Then:

(1) $A\bowtie I$ is an SFT ring.
(2) $A\bowtie I$ is not coherent.

Proof. 1) By Corollary 3.6 since A is an SFT ring.
2) $A\bowtie I$ is not coherent by [14, Theorem 4.1.5] since A is a module retract of $A\bowtie I$ and A is not coherent. □

Example 3.9. Let A be a non-coherent SFT ring, E an A-module, $B := A\propto E$ be the trivial ring extension of A by E, $f : A\to B$ be the canonical homomorphism ($f(a) = (a, 0)$) and set $J := 0\propto E$. Then:

(1) $A\bowtie^f J$ is an SFT ring.
(2) $A\bowtie^f J$ is not coherent.

Proof. 1) By Theorem 3.1 and [4, Theoreme 3.1] since $f(A) + J = B(= A \propto E)$.
2) $A \bowtie^f J$ is not coherent by [14, Theorem 4.1.5] since A is a module retract of $A \bowtie^f J$ and A is not coherent. □

References

1. D.D. Anderson; *Commutative rings*, in: Jim Brewer, Sarah Glaz, William Heinzer, Bruce Olberding (Eds.), Multiplicative Ideal Theory in Commutative Algebra: A tribute to the work of Robert Gilmer, Springer, New York, 2006, pp. 1-20.
2. J. T. Arnold, *Krull dimension in power series rings*, Trans. Amer. Math. Soc. **1**77 (1973), 299-304.
3. J. T. Arnold, *Power series rings with finite Krull dimension*, Indiana J. Math. **3**1(6), (1982), 897-911.
4. C. Bakkari; *Armendariz and SFT Properties in Subring Retracts*, Mediterr. J. Math, **6**2009, pp. 339-345.
5. M.B. Boisen and P.B. Sheldon; *CPI-extension: Over rings of integral domains with special prime spectrum*, Canad. J. Math. **29** (1977), 722-737.
6. M. Chhiti and N. Mahdou; *Some homological properties of amalgamated duplication of a ring along an ideal*, Bulletin of the Iranian Mathematical Society (to appear).
7. J. Coykendall, *The SFT property does not imply finite dimension for power series rings*, Journal of Algebra 256 (2002), 85-96.
8. J.L. Dorroh; *Concerning adjunctions to algebras*, Bull. Amer. Math. Soc. 38 (1932), 85-88.
9. M. D'Anna, C. A. Finacchiaro, and M. Fontana; *Amalgamated algebras along an ideal*, Comm Algebra and Aplications, Walter De Gruyter (2009), 241–252.
10. M. D'Anna, C. A. Finacchiaro, and M. Fontana; *Properties of chains of prime ideals in amalgamated algebras along an ideal*, J. Pure Applied Algebra **214**(2010), 1633-1641.
11. M. D'Anna; *A construction of Gorenstein rings*; J. Algebra **306**(2) (2006), 507-519.
12. M. D'Anna and M. Fontana; *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. **45**(2) (2007), 241-252.
13. M. D'Anna and M. Fontana; *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6**(3) (2007), 443-459.
14. S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics, 1371. Springer-Verlag, Berlin, 1989.
15. B.G. Kang and M.H. Park, *SFT stability via power series extensions over Prüfer domains*, Manuscripta Math., Springer-Verlag **1**22 (2007), 353-363.
16. M. Nagata; *Local Rings*, Interscience, New York, 1962.
17. M. H. Park; *Krull dimension of power series rings over a globalized pseudo-valuation domain*, J. Pure Applied Algebra **214**(2010), 862-866.
18. C. C. Weibel, *An introduction to homological algebra*, Cambridge University Press, United Kingdom, (1994).

Khalid Louartiti, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.

E − mail address : *lokha*2000*@hotmail.com*

Najib Mahdou, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.

E − mail address : *mahdou@hotmail.com*